Calc Medic Important Ideas for Unit 1: Intro to Calculus

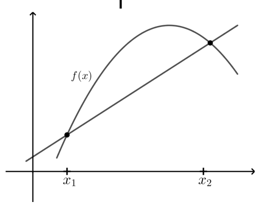
Introducing Calculus: Can Change Occur at an Instant? (Activity: A Wonder-fuel Intro to Calculus)

- Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

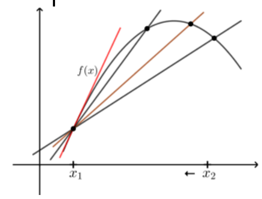
Important Ideas:

Average rate of change vs **Instantaneous rate of change**

Average rate of change
- over a measured interval
- $\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$
- slope of secant line between two distinct points



Instantaneous rate of change
- at a moment or instant in time
- $\frac{\Delta y}{\Delta x}$ as $\Delta x \to 0$
- $\frac{f(x+h) - f(x)}{(x+h) - x}$ as $h \to 0$
- slope of secant line as $x_2 \to x_1$



Defining Limits and Using Limit Notation (Activity: Can You Shoot Free Throws Like Nash?)

- Represent and interpret limits analytically using correct notation, including one-sided limits
- Estimate limits of functions using graphs or tables

Important Ideas: A limit is an intended *y*–value.

Limit notation has 3 parts:

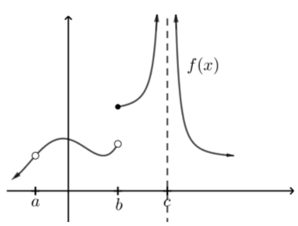
$$\lim_{x \to a} f(x) \leftarrow \text{function}$$

"*x* approaches *a*"

What does it mean for a limit to exist?

$$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L \leftrightarrow \lim_{x \to a} f(x) = L$$

- ❖ Limits from left and right must match!
- ❖ *L* is a finite number



$\lim_{x \to a} f(x)$ exists because $= \lim_{x \to a^+} f(x)$

$\lim_{x \to b} f(x)$ DNE because $\lim_{x \to b^-} f(x) \neq \lim_{x \to b^+} f(x)$

$\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ DNE because $f(x) \to \infty$

Using Algebraic Approaches to Evaluate Limits (Activity: Contestants, Can You Solve This Limit?)

- Use limit properties to determine the limits of functions
- Use algebraic manipulations to determine the limits of functions

Important Ideas:

Strategies to evaluate limits:

- direct substitution (try first)
- memorized forms $\left(\frac{\sin x}{x} \text{ or } \left|\frac{|x|}{x}\right| \dots\right)$
- Factor, simplify, substitute
- algebraic manipulation (multiply by a form of 1)
- graphs or tables
- limit properties to simplify functions

$\frac{k}{0} = \begin{cases} +\infty, k \neq 0, k > 0 \\ -\infty, k \neq 0, k < 0 \end{cases}$

$\frac{0}{k} = 0, k \neq 0$

$\frac{0}{0}, \frac{\infty}{\infty}$ are indeterminate!

$\frac{k}{\infty} = 0$

① $\lim\limits_{x \to a} c = c$

② $\lim\limits_{x \to a} c\, f(x) = c \lim\limits_{x \to a} f(x)$

③ $\lim\limits_{x \to a} (f(x) \pm g(x)) = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$

④ $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$

⑤ $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ only if $\lim\limits_{x \to a} g(x) \neq 0$

Introduction to Squeeze Theorem (Activity: How Many Coffee Beans Are In The Jar?)

- Develop an understanding of bounding values and bounding functions
- Confirm the hypotheses of the Squeeze Theorem (Sandwich Theorem, Pinching Theorem, etc.) and use the theorem to justify a limit result

Important Ideas:

Conditions: If $f(x), g(x),$ and $h(x)$ are continuous functions on some interval containing a, and $g(x) \leq f(x) \leq h(x)$ on that interval, and if $\lim\limits_{x \to a} g(x) = \lim\limits_{x \to a} h(x) = L$,

Conclusion: then by the Squeeze Theorem $\lim\limits_{x \to a} f(x) = L$

How to Justify a limit with the Squeeze Theorem

① Verify both conditions
② Identify upper bound and lower bound functions
③ Evaluate limits of upper and lower bound functions
④ Make a conclusion about original function's limit using the Squeeze Theorem

Continuity and Discontinuity (Activity: Soul Mates at Starbucks)

- Justify conclusions about continuity at a point using the definition.
- Determine intervals over which a function is continuous.

Important Ideas:

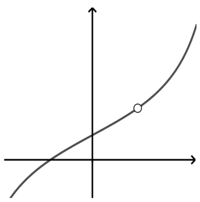
A function $f(x)$ is continuous at $x = a$ if

① $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$ (a finite #) "left and right limits match"

② $\lim_{x\to a} f(x) = f(a)$ "limit matches *y*-value"

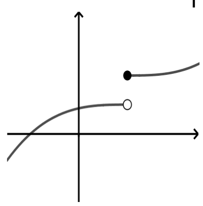
Types of Discontinuity

Removable

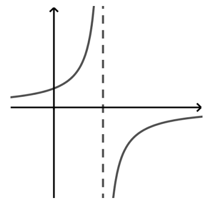


Non-removable

Jump



Infinite



Removing Discontinuities (Activity: Can This Date Be Salvaged?)

- Determine locations of removable discontinuities by graphical, numeric, or analytic methods
- Determine when and how discontinuous functions can be made continuous

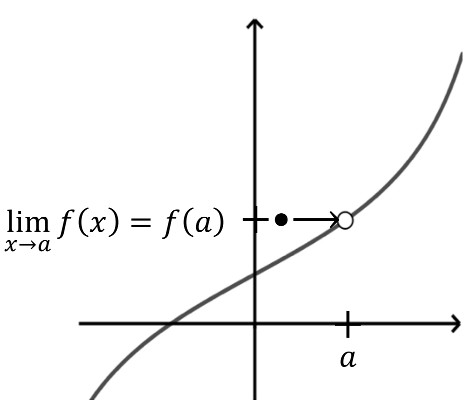
Important Ideas:

Discontinuities that occur where a limit exists can be removed by defining or redefining a point on the graph. ("patch the hole")

① Evaluate $\lim_{x\to a} f(x)$

② If the limit exists, define $f(a)$ to be $\lim_{x\to a} f(x)$

*This creates an extended function.



Limits Involving Infinity (Activity: How Much Do We Remember From School?)

- Interpret the behavior of functions using limits involving infinity.

Important Ideas:

$\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty-} f(x)$ are asking about end behavior (horizontal asymptotes)

For rational functions:

If the degree of the numerator > degree of the denominator, $\lim_{x\to\infty} f(x)$ DNE because $y \to \infty$

horizontal asymptote

If the degree of numerator < degree of the denominator, $\lim_{x\to\infty} f(x) = 0$

If the degree of the numerator = degree of the denominator,

$$\lim_{x\to\infty} f(x) = \frac{\text{leading coeffcient of numerator}}{\text{leading coeffcient of denominator}}$$

For non-rational functions, compare the dominant behavior using TFEPLC.

Tower > Factorial > Exponential > Polynomial > Logarithmic > Constant

Intermediate Value Theorem (Activity: Are You A 5-Star Uber Driver)

- Explain the behavior of a function on an interval using the Intermediate Value Theorem.

Important Ideas:

* Intermediate Value Theorem (IVT) is used to prove that a certain *y*-value <u>MUST EXIST</u>.

Condition: If a function $f(x)$ is continuous on $[a, b]$,

Conclusion: then $f(x)$ attains or "hits" every *y*-value between $f(a)$ and $f(b)$.

① Verify condition of continuity of the given function (may not be named f)

② Identify $f(a)$ and $f(b)$

③ Check that the desired value is between $f(a)$ and $f(b)$

④ Make a conclusion about using the IVT, incorporating the question stem

Since $\underbrace{f(x)}_{\text{function name}}$ is continuous on $\underbrace{[\_\_,\_\_]}_{a\ ,\ b}$ and $f(\underbrace{\quad}_{a}) = \_\_\_\_$ and $f(\underbrace{\quad}_{b}) = \_\_\_\_$, the IVT guarantees that ...

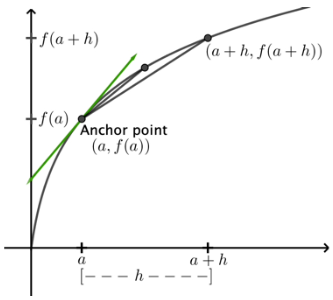
Calc Medic Important Ideas for Unit 2: Differentiation

Instantaneous Rate of Change (Activity: Can a Human Break the Sound Barrier?)

- Determine average rates of change using difference quotients
- Represent the derivative of a function as the limit of a difference quotient

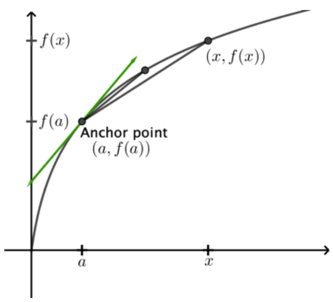
Important Ideas:

Instantaneous rate of change at $x = a$ represents the slope of the curve at $x = a$.



$$\underbrace{\lim_{h\to 0}\underbrace{\frac{f(a+h)-f(a)}{h}}_{\text{avg ROC}}}_{\text{instantaneous ROC}}$$

Definition 1



$$\underbrace{\lim_{x\to a}\underbrace{\frac{f(x)-f(a)}{x-a}}_{\text{avg ROC}}}_{\text{instantaneous ROC}}$$

Definition 2

Defining the Derivative (Activity: The Making of a Slopes Graph)

- Understand that the derivative is itself a function that outputs the slope of the curve at *any* point on the original function.

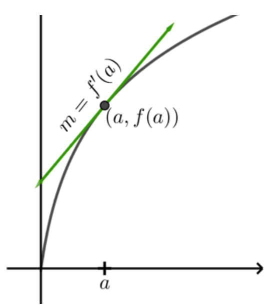
Important Ideas:

① A derivative is a function, $f'(x)$, that gives the slope of the curve at any *x*-value on $f(x)$.

② Notation for the derivative:

$$y' \qquad \frac{dy}{dx} \qquad \frac{d}{dx}y \qquad f'(x) \qquad \frac{df}{dx}$$

③ A tangent line touches the curve at one point and shares the slope of the curve.
Equation of the tangent line: $y - f(a) = f'(a)(x - a)$



Continuity and Differentiability (Activity: Is This Rollercoaster Safe to Ride?)

- Estimate the derivative at a point using graphs or tables
- Explain the relationship between differentiability and continuity
- Justify how a continuous function may fail to be differentiable at a point in its domain

Important Ideas:

Differentiable functions at $x = a$ must satisfy both conditions:

I. $f(x)$ is continuous at $x = a$

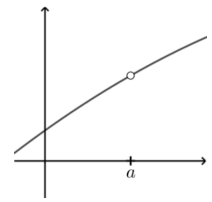
AND

II. $\lim_{x \to a^-} f'(x) = \lim_{x \to a^+} f'(x)$

*slopes must match from both sides!

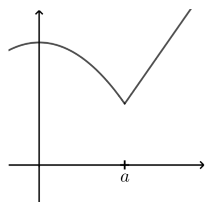
Non-differentiable functions at $x = a$:

* $f(x)$ is not continuous



OR

* slopes from both sides do not match



Continuity does not imply differentiability

Derivative Shortcuts (Activity: Is There a Shortcut?)

- Calculate derivatives of familiar functions

Important Ideas:

The derivative of a constant:	$\frac{d}{dx}c = 0$
The derivative of a constant multiplier:	$\frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = c \cdot f'(x)$
Power Rule:	$\frac{d}{dx}x^n = nx^{n-1}$ for $n \neq 0$
The derivative of a sum or difference:	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) \pm g'(x)$

Derivatives of Sin x and Cos x (Activity: Toothpick Tangents)

- Calculate the derivatives of sin x and cos

Important Ideas:

If $f(x) = \sin x$
then $f'(x) = \cos x$

If $g(x) = \cos x$
then $g'(x) = -\sin x$

Derivatives of e^x and ln x (Activity: Toothpick Tangents (Part 2))

- Calculate derivatives of familiar functions

> Important Ideas:
>
> If $f(x) = e^x$
> then $f'(x) = e^x$
>
> If $g(x) = \ln x$
> then $g'(x) = \frac{1}{x}$
> "the reciprocal of x"

The Product Rule (Activity: How Fast is Snapchat?)

- Calculate derivatives of products of differentiable functions
- Use the product rule in association with other derivative rules

> Important Ideas:
>
> Let $h(x) = f(x)g(x)$ then $h'(x) = f(x)g'(x) + g(x)f'(x)$
>
> * Some products can be simplified to avoid using the product rule.
>
> * Some functions need to be rewritten so they appear as a product of 2 functions.

Using the Quotient Rule (Activity: Divide and Conquer)

- Use the quotient rule to find derivatives of quotients of differentiable functions
- Simplify the differentiation process by choosing the correct derivative rules

> Important Ideas:
>
> For differentiable functions $f(x)$ and $g(x)$, and $g(x) \neq 0$
>
> $$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$
>
> Check results with Math 8
> Check results by comparison to Product Rule
> Simplify original rational function, if possible

Derivatives of Trig Functions (Activity: Tangents for Trig Functions)

- Calculate derivatives of products of differentiable functions
- Use identities to rewrite tangent, cotangent, secant, and cosecant functions and then apply derivative rules to find formulas for their derivatives
- Use the rules for derivatives of trigonometric functions in association with other derivative rules

Important Ideas:

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

Memorize these formulas!

Simplify or rewrite the original functions, if possible

Calc Medic Important Ideas for Unit 3
Differentiating Composite, Implicit, and Inverse Functions

The Chain Rule (Activity: How is Lindt Chocolate Made?)

- Calculate derivatives of compositions of differentiable functions

Important Ideas:

Composite Functions

$$y = f\big(g(x)\big)$$

$$y = \overbrace{f}^{\text{outside function}} \underbrace{(g(x))}_{\text{inside function "}\blacksquare\text{"}}$$

Chain Rule is used for finding derivatives of composite functions.

$$y' = g'(x) \cdot f'\big(g(x)\big)$$

$$\frac{dy}{dx} = \frac{d\blacksquare}{dx} \cdot \frac{dy}{d\blacksquare}$$

Implicit Differentiation (Activity: The Tangent Line Problem (Revisited))

- Find the derivative of implicitly defined functions

Important Ideas:

① Implicit functions are those where the dependent variable (*y*) is not isolated on one side of the equation. Example: $x^2 + xy - y^2 = 1$

② Steps for differentiating an implicitly defined function:

1) Differentiate both sides of the equation with respect to *x*.
2) Apply the chain rule to all terms with *y* in them.
3) Collect all terms with $\frac{dy}{dx}$ on one side of the equation.
4) Factor out $\frac{dy}{dx}$.
5) Solve for $\frac{dy}{dx}$ by dividing.

③ To find $\frac{d^2y}{dx^2}$, the 2nd derivative, repeat the process and substitute the function for $\frac{dy}{dx}$.

Derivatives of Inverse Functions (Activity: What's Your Slope?)

- Calculate derivatives of inverse functions

Important Ideas:

An inverse function is a function that reverses or "undoes" another function. If $f(a) = b$, then $f^{-1}(b) = a$

An inverse function exists if the original function is one-to-one. (passes the Horizontal Line Test)

* Slopes at inverse points are reciprocals!

If (a, b) is on the graph of f and g is the inverse of f, then

$$g'(b) = \frac{1}{f'(a)}$$

In general:

$$\frac{d}{dx}\big(g(x)\big) = \frac{1}{f'\big(g(x)\big)}$$

the reciprocal of

the derivative of f

evaluated at the inverse point

Derivatives of Inverse Trigonometric Functions (Activity: Getting Triggy With It)

- Calculate derivatives of inverse trig functions

Important Ideas:

$$\frac{d}{dx}(\arcsin x) = \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arccos x) = \frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan x) = \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Calc Medic Important Ideas for Unit 4: Contextual Applications of Differentiation

Interpreting the Meaning of the Derivative in Context (Activity: A Summer Day of Calculus)

- Interpret the meaning of a derivative in context.

Important Ideas:

- The derivative represents the rate of change of the dependent variable with respect to the independent variable.

$$f'(x) = \frac{df}{dx}$$

- Units of $f'(x) = \dfrac{\text{units of } f(x)}{\text{units of } x}$
- Units of $f''(x) = \dfrac{\text{units of } f'(x)}{\text{units of } x}$
- Interpreting $y'(a)$

At $x = \underbrace{\_\_\_\_\_}_{a}$, the $\underbrace{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}_{y \text{ context}}$ is increasing/decreasing at a rate of $\underbrace{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}_{\text{value with correct units}}$

Connecting Position, Velocity and Acceleration (Activity: The Lovely Ladybug)

- Calculate rates of change in the context of straight-line motion.

Important Ideas:

① Position $= s(t)$

Velocity = speed with direction

$$v(t) = s'(t)$$

Acceleration: $a(t) = v'(t) = s''(t)$

② $v(t) = 0 \Rightarrow$ object is at rest

$v(t) > 0 \Rightarrow$ object is moving right or up

$v(t) < 0 \Rightarrow$ object is moving left or down

③ Compare the signs of velocity and acceleration.

	$a(t) > 0$	$a(t) < 0$
$v(t) > 0$	Speeding up	Slowing down
$v(t) < 0$	Slowing down	Speeding up

Rates of Change in Applied Contexts Other than Motion (Activity: How Many Shoppers on Black Friday?)

- Interpret rates of change in applied contexts.

Important Ideas:

$y'(t)$ is the rate of change of $y(t)$

$y'(t) = 0 \Rightarrow \underbrace{\_\_\_\_\_\_\_}_{y \text{ context}}$ is not changing

$y'(t) > 0 \Rightarrow \underbrace{\_\_\_\_\_\_\_}_{y \text{ context}}$ is increasing

$y'(t) < 0 \Rightarrow \underbrace{\_\_\_\_\_\_\_}_{y \text{ context}}$ is decreasing

Calculator Tips

① For rate in/rate out problems:

$Y_1 = \text{rate in}$

$Y_2 = \text{rate out}$

$Y_3 = Y_1 - Y_2$

② Always round to at least 3 decimal places!

Intro to Related Rates (Activity: Birthday Balloons)

- Calculate and interpret related rates in applied contexts

Important Ideas:

Related Rate Problems

① Draw a picture

② Write an equation that relates all the variables in the problem. (usually a volume formula or Pythagorean Theorem)

③ Take the derivative of both sides. Don't forget the chain rule!

④ Plug in known values and solve for the quantity you are after.
* Determine, based on the context, if the given rates are positive or negative.

Related Rates (Activity: "Coney" Island)

- Calculate and interpret related rates in applied contexts.

Important Ideas:

<u>Cone Problems</u>

① Relate the radius to the height using similar triangles.

② Write a volume equation only in terms of r, or only in terms of h, depending on what information is given/needed.

③ Take the derivative of both sides and solve for the desired quantity.

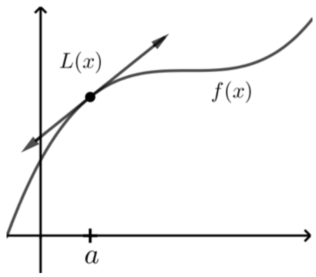
④ Use the equation from step 1 to find the rate of change of the eliminated variable.

Approximating Values of a Function Using Local Linearity and Linearization (Activity: Close Enough is Good Enough!)

- Approximate the value on a curve using the equation of a tangent line

Important Ideas:

Tangent lines can be used to approximate function values near the point of tangency.



- If $f(x)$ is concave up, $L(x)$ gives an underestimate. (tangent line is below the curve)
- If $f(x)$ is concave down, $L(x)$ gives an overestimate. (tangent line is above the curve)

$L(x) \approx f(x)$

L'Hospital's Rule (Activity: Mixed Messages)

- Interpret the rate of change at an instant in terms of average rates of change over intervals containing that instant.

Important Ideas:

Limits resulting in $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$ are considered indeterminate.

Consider the ratio of the growth rates instead!

L'Hospital's Rule:

If $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate, evaluate $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}$. If the result is still indeterminate, repeat the process by finding another derivative.

Calc Medic Important Ideas for Unit 5: Analytical Applications of Derivatives

The Mean Value Theorem (Activity: Can Calculus Get You Fined?)

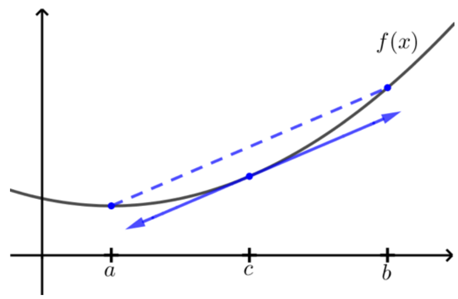
- Justify conclusions about functions by applying the MVT over an interval.

Important Ideas:

Condition: If a function $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b),

Conclusion: then there exists a value c for $a < c < b$ such that

$$\underbrace{f'(c)}_{\substack{\text{instantaneous} \\ \text{rate of change}}} = \underbrace{\frac{f(b) - f(a)}{b - a}}_{\substack{\text{average} \\ \text{rate of change}}}$$



How to prove with MVT:

① Check the conditions.

② Find the average rate of change.

③ Make the conclusion using MVT, incorporating the question stem.

Extreme Value Theorem, Absolute vs. Relative Extrema, and Critical Points (Activity: What's the Value of Apple Stock?)

- Justify conclusions about functions by applying the Extreme Value Theorem.
- Distinguish between absolute and relative extrema and critical points.

Important Ideas:

① Extreme Value Theorem:
If a function $f(x)$ is continuous on $[a, b]$ then $f(x)$must attain a maximum and minimum on $[a, b]$.

② $f(x)$ has an absolute maximum at $x = c$ if $f(c) \geq f(x)$ for all x .
$f(x)$ has a relative maximum at $x = c$ if $(c) \geq f(x)$ for all x near c .
$f(x)$ has an absolute minimum at $x = c$ if $f(c) \leq f(x)$ for all x .
$f(x)$ has a relative minimum at $x = c$ if $(c) \leq f(x)$ for all x near c .

③ Critical points are points where the derivative is 0 or undefined.

Determining Function Behavior from the First Derivative (Activity: Playing the Stock Market)

- Determine behaviors of a function based on the derivative of that function.

Important Ideas:

① $f(x)$ is increasing if $f'(x) > 0$

$f(x)$ is decreasing if $f'(x) < 0$

② First Derivative Test for Relative (Local) Extrema

- Identify the critical points ($f'(x) = 0$ or undefined)
- Make a labeled sign chart by testing values between the critical values to determine if f' is positive or negative.
- Make a conclusion

"$f(x)$ has a relative maximum of __________ at $x =$ ______ because f' changes from positive to negative"

"$f(x)$ has a relative minimum of ___________ at $x =$ ______ because f' changes from negative to positive"

Using the Candidates Test to Determine Absolute Extrema (Activity: Are You a Stock Market Master?)

- Justify conclusions about the behavior of function based on its derivative.

Important Ideas:

Finding absolute (global) maxima and minima

x	$f(x)$

① Find and list all the critical values and endpoints

② Compare the values of the function at all these locations (make a table!)

③ Write a conclusion

"$f(x)$ has an absolute maximum of __________ at $x =$ ______ "

"$f(x)$ has an absolute minimum of ___________ at $x =$ ______ "

* Must refer to Candidates Test in justification

Analyzing Function Behavior with the Second Derivative (Activity: How Fast Does the Flu Spread?)

- Justify conclusions about the behavior of function based on its second derivative

Important Ideas:

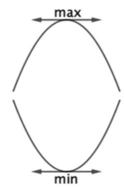
$f''(x)$ tells us how $f'(x)$ (the slopes of f) are changing.

$f''(x) > 0 \Rightarrow f'(x)$ is increasing $\Rightarrow f(x)$ is concave up.
$f''(x) < 0 \Rightarrow f'(x)$ is decreasing $\Rightarrow f(x)$ is concave down.

$f''(x) = 0$ or undefined and changes signs
$\Rightarrow f'(x)$ has a relative maximum or minimum $\Rightarrow f(x)$ has a point of inflection

2nd Derivative Test (using concavity to determine if there is a relative max or min)

- $f(x)$ has a relative maximum at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$.
- $f(x)$ has a relative minimum at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.



Optimization (Activity: Canalysis)

- Use derivatives to solve optimization problems.
- Interpret maximums and minimums in applied contexts.

Important Ideas:

Optimization is about finding a maximum or minimum in applied contexts.

① Write an equation for the quantity that is to be maximized or minimized. (volume, area, cost, distance, etc.)

② Use the constraints to find relationships between the variables.

③ Rewrite your equation with only one variable.

④ Use the 1st and 2nd derivative tests to find critical values and extrema. (or Candidates Test)

Exploring Behaviors of Implicit Relations (Activity: What About Us?)

- Determine critical points of implicit relations.
- Justify conclusions about the behavior of an implicitly defined relation based on evidence from its derivatives.

Important Ideas:

Implicit differentiation can be used to find the 1st and 2nd derivatives of relations.

Critical points are the *x* and *y* values where $\frac{dy}{dx} = 0$ or is undefined. *Make sure they satisfy the original equation!

A curve is increasing when $\frac{dy}{dx} > 0$ and decreasing when $\frac{dy}{dx} < 0$. (Specify *x* and *y* values!)

A curve is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. (Specify *x* and *y* values!)

Calc Medic Important Ideas for Unit 6: Integration and Accumulation of Change

Exploring Accumulation of Change (Activity: How Much Snow Is On Janet's Driveway?)

- Interpret the meaning of area under a rate of change function in context.

Important Ideas:

① Accumulated quantity $= \text{Rate1} \cdot \Delta t_1 + \text{Rate2} \cdot \Delta t_2 + \cdots + \text{Rate } n \cdot \Delta t_n$

② The accumulation of a quantity is represented by the area underneath its derivative curve

③ Units for area underneath a rate of change curve:
Units of rate of change • Units of the independent variable

④ Area above the x-axis is positive area ⇒ positive accumulation ⇒ quantity is increasing
Area below the x-axis is negative area ⇒ negative accumulation ⇒ quantity is decreasing

Approximating Areas with Riemann Sums (Activity: Fast and Curious)

- Approximate area under a curve using geometric and numerical methods

Important Ideas:

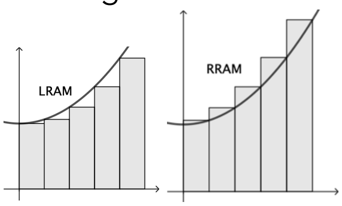
To approximate the area on $[a, b]$ with *n* equal subdivisions use $\frac{b-a}{n}$ as the width of each rectangle.

To find the height of each rectangle:

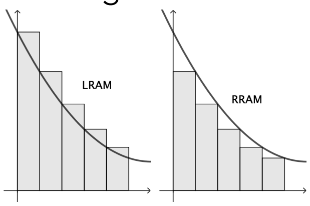
- LRAM – Evaluate the function at the left endpoint of each interval
- RRAM – Evaluate the function at the right endpoint of each interval
- MRAM – Evaluate the function at the midpoint of each interval

Total area/accumulation $= f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots + f(x_n)\Delta x_n$

If a function is increasing:
LRAM gives an underestimate
RRAM gives an overestimate



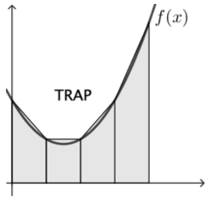
If a function is decreasing:
LRAM gives an overestimate
RRAM gives an underestimate



Approximating Areas with Trapezoids (Activity: Fast and Curious (Part 2))

- Approximate area under a curve using geometric and numerical methods

Important Ideas:



Evaluate function at right endpoint

$$\text{Area of a trapezoid} = \frac{(b_1+b_2)}{2}\cdot h$$

Evaluate function at left endpoint

Δx

Area under $f(x) \approx$ sum of trapezoids

A trapezoidal sum gives the average of LRAM and RRAM values.

Riemann Sums, Summation Notation, and Definite Integrals (Activity: How Confident Are You?)

- Interpret and represent an infinite Riemann sum as a definite integral

Important Ideas:

To find the exact area under a curve $f(x)$ over the interval $[a, b]$, use infinitely many rectangles of infinitely small widths.

Summation Notation:

Number of rectangles

$$\lim_{n\to\infty}\sum_{i=1}^{n}\underbrace{f(x_i)}_{\text{height of rectangle}}\ \underbrace{\Delta x}_{\text{width of rectangle}}$$

Integral Notation:

upper limit of integration

$$\int_a^b \underbrace{f(x)}_{\text{integrand}}\ \underbrace{dx}_{\text{variable of integration}}$$

lower limit of integration

The Fundamental Theorem of Calculus and Accumulation Functions (Activity: Under Cover)

- Represent accumulation functions using definite integrals.
- Find derivatives of accumulation functions.

Important Ideas:

An accumulation function outputs the area under the curve form some starting value to x (the input).

Independent variable

rate of accumulation

$$F(x) = \int_c^x f(t)dt$$

starting value

FTC (Part 1): Rate of change of an accumulation function:

$$\frac{d}{dx}\int_c^x f(t)dt = \frac{d}{dx}F(x) = f(x)$$

Applying Properties of Definite Integrals (Activity: #2020 Goals)

- Calculate a definite integral using areas and properties of definite integrals.

Important Ideas:

For $a < b < c$

① $\displaystyle\int_a^c f(t)dt = \int_a^b f(t)dt + \int_b^c f(t)dt$

② $\displaystyle\int_b^a f(t)dt = -\int_a^b f(t)dt$

③ $\displaystyle\int_a^b [f(t) \pm g(t)]dt = \int_a^b f(t)dt \pm \int_a^b g(t)dt$

④ $\displaystyle\int_a^b kf(t)dt = k\int_a^b f(t)dt$

⑤ $\displaystyle\int_a^a f(t)dt = 0$

The FTC and Definite Integrals (Activity: Go Figure)

- Evaluate definite integrals analytically using the Fundamental Theorem of Calculus.

Important Ideas:

$$\int_a^b f(x)dt = F(x)\Big|_a^b = F(b) - F(a)$$

where $F(x)$ is the antiderivative of $f(x)$ $\quad (F'(x) = f(x))$

$$\underbrace{F(b)}_{\text{Final Value}} = \underbrace{F(a)}_{\text{Initial Value}} + \underbrace{\int_a^b f(x)dt}_{\text{Accumulated Change}}$$

Finding Antiderivatives and Indefinite Integrals: Basic Rules and Notation (Activity: A Match Made in Heaven)

- Determine antiderivatives of functions and indefinite integrals, using knowledge of derivatives.

Important Ideas:

① Given a function $f(x)$, the most general antiderivative of $f(x)$ is given by

Indefinite Integral (no upper & lower bounds $\rightarrow$ $\displaystyle\int f(x)dx = F(x) + C$ where $F'(x) = f(x)$

Constant of Integration

② Antiderivative Rules

$\displaystyle\int kdx = kx + C$

$\displaystyle\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

$\displaystyle\int \frac{1}{x}dx = \ln|x| + C$

$\displaystyle\int e^x dx = e^x + C$

$\displaystyle\int \sin x\, dx = -\cos x + C$

$\displaystyle\int \cos x\, dx = \sin x + C$

Integration using Substitution (Activity: Which One Doesn't Belong?)

- Use u-substitution to find antiderivatives of composite functions

Important Ideas:

$$\left.\begin{array}{l}\frac{d}{dx}\big(f(u(x))\big) = u'(x)\cdot f'(u(x)) \\ \int \underbrace{u'(x)}_{\text{derivative of inner function}}\cdot f'(\underbrace{u(x)}_{\text{inner function}})dx = f(u(x)) + C\end{array}\right\}$$ Integration with substitution reverses the chain rule!

① Set *u* = inner function

② Find $\frac{du}{dx}$ and solve for du

③ "Correct" the integrand so it matches du exactly. (you can only multiply by a scalar form of 1)

④ Rewrite the entire integral in terms of *u*

⑤ Find the antiderivative.

⑥ For indefinite integrals, plug in the expression for *u*.

Riemann Sums, Summation Notation, and Definite Integral Notation (Activity: Returning to Riemann)

- Interpret and represent an infinite Riemann sum as a definite integral

Important Ideas:

A definite integral can be represented by an infinite Riemann sum.

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f(x_i)}_{\substack{\text{height of}\\ \text{i}^{\text{th}}\text{ rectangle}}} \underbrace{\Delta x}_{\substack{\text{width of}\\ \text{rectangle}}} \qquad \text{where } n \text{ is the \# of partions/rectangles}$$

Assuming equal partitions:

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\cdot\Delta x \qquad \int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{f\left(a + i\left(\frac{b-a}{n}\right)\right)}_{\text{height}}\cdot\underbrace{\left(\frac{b-a}{n}\right)}_{\text{width}}$$

Calc Medic Important Ideas for Unit 7: Differential Equations

Modeling Differential Equations and Verifying Solutions (Activity: How Long Does Coffee Stay Hot?)

- Interpret differential equations given in context
- Verify solutions to differential equations

Important Ideas:

A differential equation is any equation that contains a derivative expression.

$\frac{dy}{dx}$ is a first order derivative, $\frac{d^2y}{dx^2}$ is a second order derivative, …

Differential equations may contain the original function or be written in terms of the independent variable only.

If a rate is proportional to the current quantity, then $\frac{dy}{dx} = ky$ where "k" is the constant of proportionality.

Slope Fields (Activity: Seeing is Believing)

- Create slope fields
- Estimate solutions to differential equations

Important Ideas:

Slope fields are a graphical representation of a differential equation that allows us to visualize the family of solutions curves.

Making a slope field:

- Calculate the slope at various ordered pairs (points).
- Plot short line segments using the slopes at the various points.

A solution curve will follow the trend of the slopes and must pass through the initial condition point if given.

If $\frac{dy}{dx}$ is undefined, do not draw a slope there.

Finding General Solutions using Separation of Variables (Activity: Are you a Solution Seeker?)

- Determine general solutions to differential equations

Important Ideas:

Use separation of variables to solve a first order differential equation of the form $\frac{dy}{dx} = f(x) \cdot g(y)$

① Separate (Move all terms with *y* to one side and all terms with *x* to the other)

② Integrate (Don't forget +*C* !)

③ Isolate (Get *y* by itself, watch out for $\pm$ solutions)

Finding Particular Solutions using Initial Conditions and Separation of Variables (Activity: How many Sea Lions are on Elliott Bay?)

- Determine particular solutions to differential equations

Important Ideas:

① Some particular solutions can't be written explicitly and must be defined by integrals.

If $\frac{dy}{dx} = f(x)$ and $\underbrace{F(a)}_{\substack{\text{antiderivative} \\ \text{of } f(x)}} = y_0$ then $F(x) = y_0 + \int_a^x f(t)dt$ is a solution.

② Finding particular solutions requires 2 extra steps:

Separate Integrate **Solve for *C*** Isolate **Select**

Remember SISIS !

Exponential Models with Differential Equations (Activity: How Fast is the Coronavirus Spreading?)

- Interpret the meaning of a differential equation and its variables in context

Important Ideas:

$$\frac{dy}{dx} = ky \Rightarrow \left\{\begin{matrix}\text{rate of change} \\ \text{of a quantity is} \\ \text{proportional to} \\ \text{the quantity}\end{matrix}\right\} \Rightarrow \text{exponential growth/decay mode} \quad \left\{\begin{matrix}k > 0 \Rightarrow \text{growth} \\ k < 0 \Rightarrow \text{decay}\end{matrix}\right\}$$

$$\frac{dy}{dx} = ky \quad \text{has solutions of the form } y = y_0 e^{kt}$$

initial condition when $t = 0$

Calc Medic Important Ideas for Unit 8: Applications of Integration

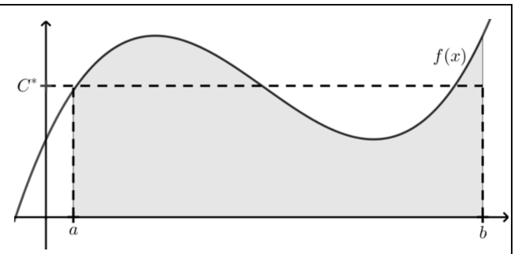
Average Value of a Function (Activity: Finding the Perfect Rectangle)

- Determine the average value of a function using definite integrals

Important Ideas:

The average value of a continuous function $f(x)$ on the interval $[a, b]$ is the height of the rectangle that encompasses the same area as the area under the curve.

$$\underbrace{C^*(b-a)}_{\substack{\text{area of}\\ \text{rectangle}}} = \underbrace{\int_a^b f(x)dx}_{\substack{\text{area under}\\ \text{the curve}}} \Rightarrow C^* = \frac{1}{b-a}\int_a^b f(x)dx$$



Connecting Position, Velocity, and Acceleration using Integrals (Activity: Whitney's Bike Ride)

- Determine values for positions and rates of change using definite integrals in problems involving rectilinear motion

Important Ideas:

Given velocity $v(t)$ and position $s(t)$

$$\int_a^b v(t)dt = \underbrace{s(b)-s(a)}_{\substack{\text{change in position}\\ \text{net distance}\\ \text{displacement}}} \quad \text{or} \quad \underbrace{s(b)}_{\substack{\text{final}\\ \text{position}}} = \underbrace{s(a)}_{\substack{\text{initial}\\ \text{postion}}} + \underbrace{\int_a^b v(t)dt}_{\substack{\text{change in}\\ \text{position}}}$$

$$\left.\begin{matrix}\text{Total distance}\\ \text{traveled on } [a,b]\end{matrix}\right\} = \int_a^b |v(t)|dt$$

Using Accumulation and Definite Integrals in Applied Contexts (Activity: How many People are at the Met?)

- Interpret the meaning of a definite integral in accumulation problems
- Determine net change using definite integrals in applied contexts

Important Ideas:

Given a quantity $y(t)$ and its rate of change $y'(t)$

① $$\underbrace{\int_a^b y'(t)dt}_{\text{integral of a rate of change}} = \underbrace{y(b) - y(a)}_{\text{net change in quantity}}$$

② The accumulation equation:

$$y(t) = y(a) + \int_a^t y'(x)dx$$

③ Sometimes $y'(t)$ consists of a rate in and a rate out.

Calculator tips:

$Y_1 = \text{rate in}$

$Y_2 = \text{rate out}$

$Y_3 = Y_1 - Y_2$

Finding Areas between Curves Expressed as Functions of x (Activity: How Rich are the Top 1%?)

- Calculate areas in the plane using the definite integral.

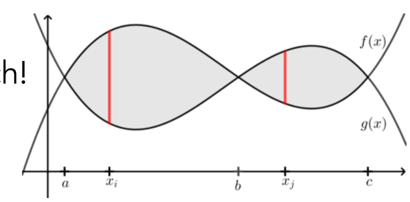
Important Ideas:

- Area between $f(x)$ and $g(x)$ on $[a, b]$ when $f(x) \geq g(x)$ is given by

$$A = \int_a^b \left[\underbrace{f(x)}_{\text{upper curve}} - \underbrace{g(x)}_{\text{lower curve}} \right] dx$$

- The region <u>must</u> be bounded.
- Area is ALWAYS positive
- Sometimes the upper and lower functions switch!

$$A = \int_a^b [f(x) - g(x)]dx + \int_b^c [g(x) - f(x)]dx$$



Finding Areas between Curves Expressed as Functions of y (Activity: How Do You Build a Deck?)

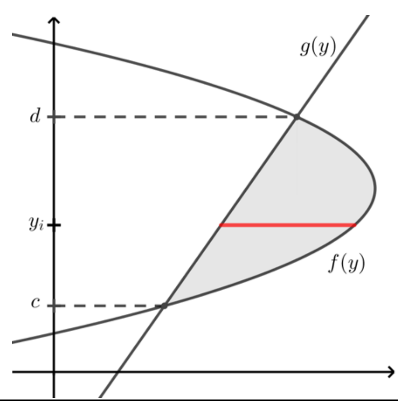
- Calculate areas in the plane using the definite integral.

Important Ideas:

If the upper curve requires 2 or more definitions, consider using a right curve and a left curve (horizontal rectangles)

$$\left.\begin{matrix}\text{Area of} \\ \text{region } R\end{matrix}\right\} = \int_{\underset{y\text{-value}}{c}}^{\overset{y\text{-value}}{d}} [f(y) - g(y)] \underbrace{dy}_{\text{width}}$$

Left curve: $g(y)$ — Right curve: $f(y)$



Volume using the Disc Method (Activity: What is the Volume of a Pear?)

- Calculate volumes of solids of revolution using definite integrals

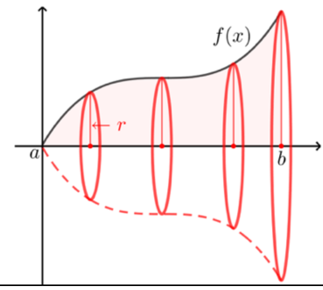
Important Ideas:

Some solids are generated by revolving 2-d regions around an axis of revolution.

If the axis of revolution (AOR) is a boundary of the region, then each slice is a disk.

$$V = \int_a^b \pi r^2 dx = \pi \int_a^b r^2 dx \text{ or } V = \pi \int_a^b r^2 dy$$

How to find the radius length: Draw a segment from AOR to the curve, then subtract "upper – lower" or "further right – closer".



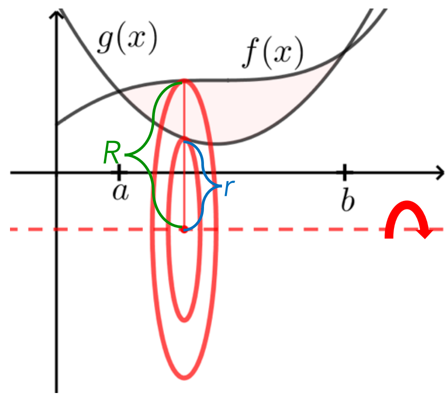
Volume using the Washer Method (Activity: What's the Volume of a Bagel?)

- Calculate volumes of solids of revolution using definite integrals

Important Ideas:

Use the Washers Method when there is a space/gap between the region and the axis of revolution.

1) Draw sample radii from the axis of revolution to the boundaries of the region.
2) Write an expression for R and r. Think "upper – lower" or "further right – closer".
3) Set-up an integral. Think (area of larger circle – area of smaller circle) times thickness.

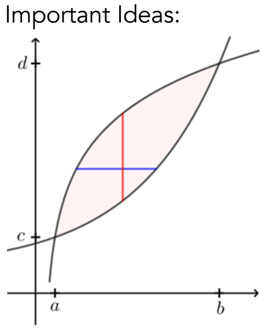


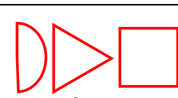
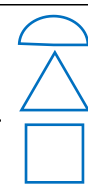
$$\underbrace{\int_a^b \left[\underbrace{\pi\left(R(x)\right)^2}_{\text{larger area}} - \underbrace{\pi\left(r(x)\right)^2}_{\text{smaller area}}\right] \underbrace{dx}_{\text{width}}}_{\text{horizontal AOR } y=k} = \pi \int_a^b \left[\left(R(x)\right)^2 - \left(r(x)\right)^2\right] dx \text{ or } \underbrace{\pi \int_c^d [(R(y))^2 - (r(y))^2] dx}_{\text{vertical AOR } x=k}$$

Volumes With Cross Sections (Activity: The Best Thing Since Sliced Bread)

- Calculate volumes of solids with known cross sections using definite integrals

Important Ideas:



1) Draw a sample cross-section.  or 
2) Write an expression for the area of one cross-section.
3) Set-up an integral.

$$\overbrace{\int_a^b A(x)dx}^{\substack{\text{Perpendicular} \\ \text{to } x-\text{axis}}} \text{ or } \overbrace{\int_c^d A(y)dy}^{\substack{\text{Perpendicular} \\ \text{to } y-\text{axis}}}$$